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# MAKING ACTION-ANGLE DISC MODELS FOR GAIA

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**Abstract.** I describe dynamical modelling of the Milky Way using action-angle coordinates. I explain what action-angle coordinates are, and what progress has been made in the past few years to ensuring they can be used in reasonably realistic Galactic potentials. I then describe recent modelling efforts, and progress they have made in constraining the potential of the Milky Way and the local dark matter density.

## 1 Introduction

When available, action-angle variables are the most convenient way of describing orbits in a given gravitational potential  $\Phi$ . The three actions ( $\mathbf{J}$ ) are constants of motion, and therefore label an orbit. They are the conjugate momenta of the angle coordinates  $\boldsymbol{\theta}$ . Since  $\mathbf{J}$  is constant, basic Hamiltonian mechanics allows us to write that

$$-\frac{\partial H}{\partial \boldsymbol{\theta}} = \dot{\mathbf{J}} = 0. \quad (1.1)$$

This means that  $H$  must be independent of  $\boldsymbol{\theta}$ . The derivatives of  $H$  are therefore also clearly independent of  $\boldsymbol{\theta}$ , so we can again use basic Hamiltonian mechanics to write

$$\dot{\boldsymbol{\theta}} = \frac{\partial H}{\partial \mathbf{J}} = \boldsymbol{\Omega}(\mathbf{J}) \quad (1.2)$$

where  $\boldsymbol{\Omega}$  is known as the frequencies *and is independent of  $\boldsymbol{\theta}$* . Therefore the angles of a particle on a given orbit (i.e. fixed  $\mathbf{J}$ ) increase linearly with time – we can write this in component form as

$$\theta_i(t) = \theta_i(0) + \Omega_i(\mathbf{J})t. \quad (1.3)$$

Since the orbit is bound and regular, the position and velocity of an object is a *periodic* function of the angles. This periodicity is defined to be over  $2\pi$  (so if we increase any angle coordinate by  $2\pi$  we return to the original position).

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The components of the actions can be defined by

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{J} \cdot d\boldsymbol{\theta} = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot d\mathbf{q} \quad (1.4)$$

where the path  $\gamma_i$  is one on which  $\theta_i$  increases by  $2\pi$ . (An expansion of why the second equality is true, along with a much more detailed discussion of action-angle coordinates can be found in Binney & Tremaine 2008.) For example, a path in an axisymmetric potential that goes from  $\phi = 0$  to  $\phi = 2\pi$  (while other coordinates are unchanged) is clearly one such path, we then have

$$J_\phi = \frac{1}{2\pi} \int_0^{2\pi} p_\phi d\phi = \frac{1}{2\pi} \int_0^{2\pi} L_z d\phi = L_z, \quad (1.5)$$

where  $L_z$  is the angular momentum about the  $z$  axis.

So, in a given potential, the values  $\mathbf{J}$  label an orbit, and  $\boldsymbol{\theta}$  labels a point on that orbit.

The relationship between action-angle coordinates and the position and velocity of a particle will clearly depend on the gravitational potential in which the particle is moving. The major reason that action-angle coordinates are not widely used in astrophysics is that – for most potentials – it is not possible to determine them by simple techniques.

In a spherically symmetric potential two of the actions are simply related to the total angular momentum and the angular momentum about some preferred axis. The third action and the angles can be found by a 1D numerical integral. In the special case of the isochrone potential these integrals can be performed analytically.

This is possible because the equations of motion are separable in spherical polar coordinates in a spherically symmetric potential. The other instance where the equations of motion are separable is in the Stäckel family of potentials (e.g. de Zeeuw 1985). Again, in this case, the angles and actions can be found by simple 1D integrals – in this case in confocal ellipsoidal coordinates.

### 1.1 Finding action-angle coordinates for more realistic galactic potentials

Over the past few years there has been substantial improvement in the methods available for calculating action-angle coordinates in other potentials. In the past it was common to estimate them using the “adiabatic approximation” (e.g. Binney 2010), which is the approximation that the motion perpendicular to the Galactic plane can be decoupled from the motion parallel to the plane. The “Stäckel fudge” improves on this by approximating that the motion can be decoupled in the ellipsoidal coordinates associated with Stäckel potentials (Binney 2012b). This has now been extended to triaxial potentials (Sanders & Binney 2015a).

Other methods require the use of so-called “generating functions” to manipulate the known action-angle coordinates in a given potential (typically the isochrone potential) so that they are valid in a new potential. The “torus method”

(e.g. McMillan, Binney et al, in prep – <https://github.com/PaulMcMillan-Astro/Torus>) uses this to determine the full phase-space structure of an orbit with a given  $\mathbf{J}$  in a given potential. A new method based on an orbit integration (Sanders & Binney 2014 – <https://github.com/jlsanders/genfunc>) allows one to do similar given an initial position and velocity, rather than a value  $\mathbf{J}$ .

These methods are now publicly available. The axisymmetric “Stäckel fudge” will be made available by Binney in the near future. A version of the Stäckel fudge, and routines similar to Sanders and Binney’s orbit integration method are also available through GALPY (Bovy 2015 – <http://github.com/jobovy/galpy>).

## 2 Modelling

Distribution functions (DFs) that are a function of action alone are in equilibrium (this follows from Jeans’ theorem). Simple functional forms for the DF of a disc galaxy in equilibrium have been in use for around 5 years (Binney 2010, though the commonly used form is the altered version used by Binney & McMillan 2011). They are of the form

$$f(\mathbf{J}) = f_\phi(J_\phi) f_r(J_r, J_\phi) f_z(J_z, J_\phi) \quad (2.1)$$

where  $f_\phi$  primarily controls the radial surface density,  $f_z$  primarily controls the vertical density and velocity profile, and  $f_r$  primarily controls the radial and azimuthal velocity distributions. We take

$$f_r \propto \exp(-\kappa J_r / \sigma_r^2); \quad f_z \propto \exp(-\nu J_z / \sigma_z^2). \quad (2.2)$$

These have had substantial success in fitting the local velocity distribution and density profiles (e.g. Binney 2012a). Since they ensure consistency between the radial and azimuthal velocity distributions, initial attempts to fit the local velocity distribution were unsuccessful. This was shown to be because the peculiar velocity of the Sun differed by around  $7 \text{ km s}^{-1}$  from the value that was accepted at the time, and assumed in the initial analysis (Binney 2010).

Alternative DFs for halo-like components have also been proposed and used (Binney 2014, Posti et al. 2015). The disc DFs have been adapted to include velocity substructure (McMillan 2011, 2013), or information about chemistry (Sanders & Binney 2015b).

The biggest reason to use DFs of the form  $f(\mathbf{J})$  is that it allows one to learn about the gravitation potential that the stars are moving in. To learn anything about the potential one has to start from the approximation that the Galaxy is in equilibrium, as otherwise any set of observed stellar positions and velocities are consistent with any potential.

It is not a simple task to fit observational data about the Milky Way to these models, but a method was demonstrated on mock data (McMillan & Binney 2013, see also Ting et al. 2013). The key point recognised was that an approach based upon an orbit library was doomed to fail, as the orbit library could never hope to be sufficiently dense that each observed star had a reasonable number of orbits sampling its error ellipse. The next step was to do this for real data.

### 3 The potential of the Milky Way from RAVE data

A new study (Piffl et al. 2014) uses data from the RAVE survey to constrain the Milky Way’s potential and the local dark matter density. It does this by demanding that the derived DF and gravitational potential satisfy 3 demands: 1) It must fit binned velocity histograms taken from the RAVE survey (after allowing for uncertainties). 2) The stellar density as a function of  $z$  above the sun due to the DF must match that of the stellar component of the mass model that produces the potential. 3) This vertical density profile is also fit to a vertical density profile for the Milky Way found from the SDSS (Jurić et al. 2008).

This work was able to provide constraints on the Milky Way potential, which are described in detail in the paper. The headline result is that it found the local dark matter density (assuming an oblate or spherical halo with axis ratio  $q$ ) is

$$\rho_{\text{dm},\odot} = (0.48 \times q^{-\alpha}) \text{ GeV cm}^{-3} \quad (3.1)$$

with a systematic uncertainty of 15 per cent. The main contribution to the uncertainty is the uncertain distances to stars observed by RAVE. This is an uncertainty that Gaia will dramatically reduce.

It should also be noted that a similar study by Bovy & Rix (2013), using data from the Segue survey, found results consistent with those found by Piffl et al.

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